Microscopic model for the logarithmic size effect on the Curie point in Barabási-Albert networks

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Abstract

We found that numbers of fully connected clusters in Barabási-Albert (BA) networks follow the exponential distribution with the characteristic exponent $\kappa = 2/m$. The critical temperature for the Ising model on the BA network is determined by the critical temperature of the largest fully connected cluster within the network. The result explains the logarithmic dependence of the critical temperature on the size of the network N.

Keywords: Monte Carlo simulation, Curie temperature, cluster statistics

During the last few years studies of random, evolving networks have become a very popular research domain among physicists. A lot of efforts were put into investigation of such systems in order to recognize their structure and to analyze emerging complex dynamics. The Barabási-Albert (BA) network [1] being the subject of this paper is probably the most studied model that realizes many properties of real weblike systems. The most important feature of the model is the power-law degree distribution that is

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also a characteristic of many real networks. It was shown that this feature is closely related to processes governing system evolution. Two important ingredients of the BA model are: continuous network growth and preferential attachment. The network starts to grow from an initial cluster of m fully connected sites. Each new node that is added to the network creates m links that connect it to previously added nodes. The preferential attachment means that the probability of a new link to end up in a vertex i is proportional to connectivity of this vertex k_i.

The goal of the investigation run here was to understand the logarithmic dependence of the Curie temperature in ferromagnetic phase transition in BA network [2] on the system size. A mean field theory for this behaviour has been recently found by Bianconi [3] and independently developed by Dorogovtsev at al. [4]. Here we provide another explanation for the fact and associate it with the microscopic, internal structure of the BA networks.

We investigate the probability to occur a fully connected cluster of size s in BA networks. A fully connected s-cluster is defined as a group of s points, such that every point of the group has a connection to each of the $(s-1)$ remaining points within this group. In order to get reliable numbers n_s of fully connected s-clusters all s-subsets of the whole set of network vertices should be examined. It gives $\binom{N}{s}$ possibilities for each cluster of size s and altogether 2^N combinations for all cluster sizes. Such calculations would involve unreasonable amount of computer time. To simplify the task we noticed that each point i that has k_i nearest neighbors may be involved in $\binom{k_i}{s-1}$ fully connected clusters of size s. Because of computer time limitations we were forced to restrict our studies to nodes with degree $k_i \leq k_{max}$ where $k_{max} = 20$ or 30 and small lattices $N = 28$, 38, 53 (Fig.1). All simulations were done for BA networks with $m = 3$. Results presented at Fig.1 are exact in this sense that all possible combinations of fully connected clusters up to size $s = k_{max} + 1$ having at least one node i of degree $k_i \leq k_{max}$ were checked. Numerical simulations carried out for networks of size N show that the cluster numbers n_s follow the exponential distribution:

$$n_s = ANe^{-\kappa * s}. \tag{1}$$

At the Fig.1 numbers of such s-clusters per network node n_s/N are fitted to (1) giving

$$\begin{aligned} \ln(A) &= 3.20 \pm 0.20 \\ \kappa &= 0.70 \pm 0.05. \end{aligned} \tag{2}$$

The critical temperature for the Ising model on the fully connected lattice

that consists of s sites follows the well-known mean-field result

$$T_c = (s-1), \tag{3}$$

where $(s-1)$ is the mean number of nearest neighbors (temperature is measured in units of coupling constant over Boltzman constant J/k). Let us assume that the critical temperature for the largest fully connected cluster within the BA network is given by (3). Since fully connected clusters in BA model are not isolates lattices but they are involved in the whole network structure the equation (3) is valid only for the largest cluster. Putting $n_s = 1$ and using (1) we get the temperature as:

$$T_c = \frac{\ln(A)}{\kappa} - 1 + \frac{\ln(N)}{\kappa}. \tag{4}$$

Taking into account the result (2) for the BA model with $m = 3$ we get:

$$T_c = B\ln(N) + C, \tag{5}$$

where $B = 1.4 \pm 0.1$ and $C = 3.6 \pm 0.1$. Looking at the result obtained by Dorogovtsev at al. [4] for the whole BA network:

$$T_c = m\ln(N)/2 \tag{6}$$

and putting $m = 3$ into (6) the critical temperature of the whole network gains the form:

$$T_c = 1.5\ln(N), \tag{7}$$

that for large N fairly good agrees with (5) and (4) giving $\kappa = 2/m$. It follows that the Curie temperature of the whole BA network is closely related to the Curie temperature of the largest fully connected cluster. To destroy the magnetisation of the network one needs to destroy ferromagnetic coupling within this cluster. Our studies are consistent with the general conclusion of the paper [4] that the most connected vertices in random graphs with arbitrary degree distributions induce strong ferromagnetic correlations in their close neighborhoods at high temperatures. Such vertices probably participate in the fully connected cluster making the ordering phenomena of the whole network dependent on ordering of the largest fully connected cluster.

In summary, we presented a possible microscopic explanation for logarithmic size effect of the Curie temperature in Barabási-Albert networks. We found that the critical temperature of the whole network is determined by the critical temperature of the largest fully connected cluster within the network.

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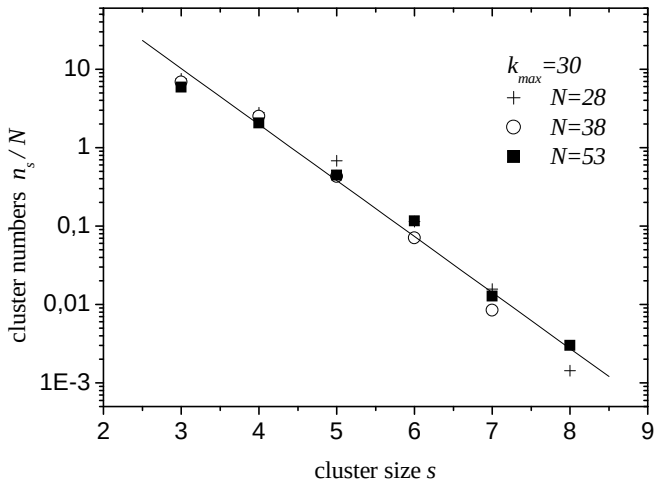

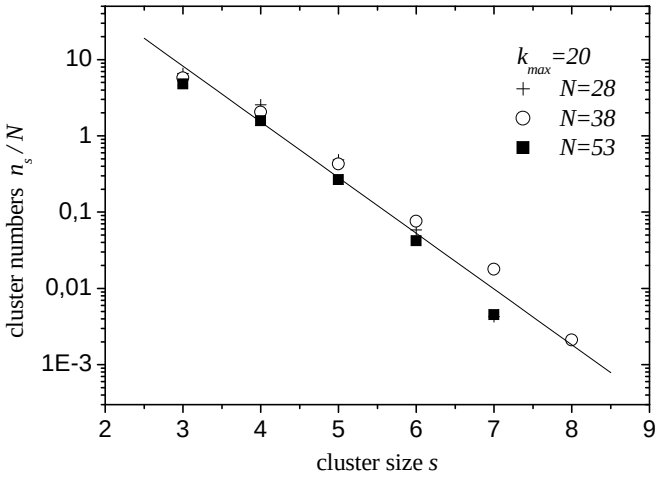

Figure 1: Distributions of fully connected clusters, $k_{max} = 30$ (a) or 20 (b) (averaged over 50 networks)